TTIC 31230, Fundamentals of Deep Learning

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CNNs: Einstein Notation

Einstein Notation

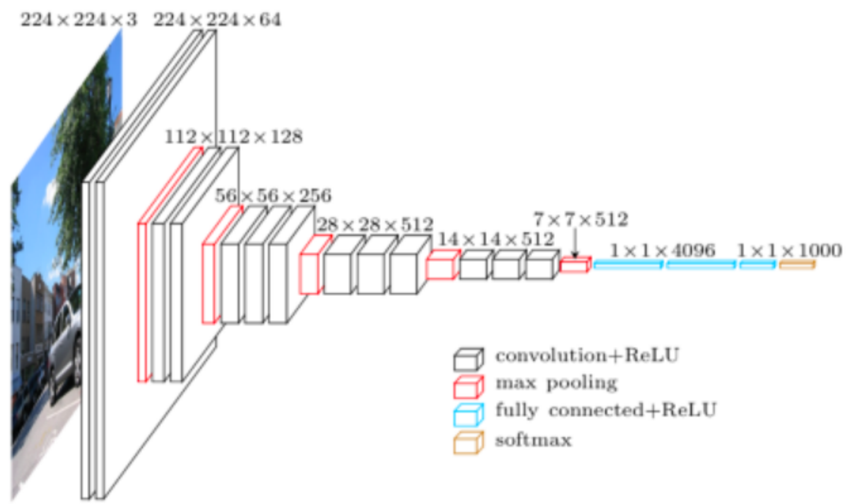
For the representation of general relativity, Einstein introduced the convention of explicitly writing all indeces of tensors where repeated indeces in a product of tensors are implicitly summed.

Writing indeces explicitly improves the clarity of the notation at the expense of not being in correspondence with framework notation. Most frameworks hide indeces.

This course will focus on conceptual understanding rather than framework implementations. For conceptual understanding Einstein notation seems preferable.

Advantages of Einstein Notation

The indeces of tensors generally have types such as “batch index”, “x coordinate”, “y coordinate” and “neuron index”.



A layer in a CNN has “shape” $L[b, x, y, n]$ where b is a batch intex, n is a nueron index, and x and y specify a spacial locotation.

Advantages of Einstein Notation

A layer in a CNN has shape $L[b, x, y, n]$.

Writing the indeces explicitily typically makes the meaning of indeces clear and avoids having to remember the somewhat arbitrary order of the indeces (the order matters for efficiency of memory access — discussed later).

Einstein Notation

We will use a modified form of Einstein notation where captial letters are used to denote slices of a tensor. For example:

- $L[b, x, y, n]$ denotes a single (scalar) number.
- $L[b, x, y, N]$ denotes a vector of neuron values.
- $L[b, X, Y, N]$ denotes the entire layer of the bth batch element.

Einstein Notation

Frameworks (and NumPy) use C-order (row-major) in larying out a tensor in memory.

The vector $L[b, x, y, N]$ denotes a continuous block of memory.

The bth batch layer $L[b, X, Y, N]$ also denotes a contiguous block of memory.

Einstein Notation

I will use repeated capital letters in a product of tensors to denote summation over those letters.

$$
\begin{aligned}
y = Wx \quad & \equiv \quad y[i] = \sum_j W[i,j]x[j] \\
& \equiv \quad y[i] = W[i,J]x[J]
\end{aligned}
$$

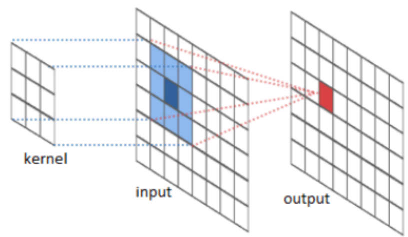
$$
\begin{aligned}
y = x^\top W \quad & \equiv \quad y[j] = \sum_i W[i,j]x[i] \\
& \equiv \quad y[j] = W[I,j]x[I]
\end{aligned}
$$

Einstein Notation for Convolution

CNNs provide a good example of the advantage of Einstein Notation.

$L[b, x, y, i]$ is the value of "neuron" i for batch element b at image position $\langle x, y \rangle$.

Convolution



$$W[\Delta x, \Delta y, i, j] \qquad L_\ell[b, x, y, i] \qquad L_{\ell+1}[b, x, y, j]$$

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$$L_{\ell+1}[b, x, y, j] \;=\; \sigma\left(\sum_{\Delta x, \Delta y, i} W[\Delta x, \Delta y, i, j]\; L_\ell[b, x + \Delta x, y + \Delta y, i] - B[j]\right)$$

$$=\; \sigma\left(W[\Delta X, \Delta Y, I, j]\; L_\ell[b, x + \Delta X, y + \Delta Y, I] - B[j]\right)$$

Types and Einstein Notation

The indeces of tensors generally have types such as a “time index”, “x coordinate”, “y coordinate”, “batch index”, or “neuron index”.

Writing a matrix as $W[T, I]$ where T is a time index and I is a feature index makes the type of the matrix W clear and clarifies the order of the indeces (disambiguates W from $W^\top$).

Writing a layer of a CNN as $L[B, X, Y, I]$ clarifies both the types and the positions of the four indeces.

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