TTIC 31230, Fundamentals of Deep Learning

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Noisy Channel RDAs

The KL term as Channel Capacity

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}} \; E_{y,z} \;\; \ln \frac{p_\Psi(z|y)}{p_\Phi(z)} \; - \ln p_\Phi(y|z)$$

$$= \underset{\Phi}{\operatorname{argmin}} \; I_{\Psi,\Phi}(y,z) + E_{y,z} \; - \ln p_\Phi(y|z)$$

The mutual information $I_{\Psi,\Phi}(y,z)$ is the channel capacity giving the **rate** of information transfer from y to z.

L_2 Distortion

$$\mathcal{L}(\Phi) = E_{y\sim\mathrm{Pop}}\; - \ln P_\Phi(\tilde{z}_\Phi(y)) + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}_\Phi(y)))$$

It is common to take

$$\begin{aligned} \mathrm{Dist}(y,\hat{y}) &= ||y-\hat{y}||^2 \qquad (L_2) \\ &= -\frac{1}{\lambda}\ln p(y|\hat{y}) + C \qquad \text{for } p(y|\hat{y}) \propto \exp(-\lambda||y-\hat{y}||^2) \end{aligned}$$

We will ignore the log density interpretation and just call this L_2 distortion.

L_1 **Distortion**

$$\mathcal{L}(\Phi) = E_{y\sim\mathrm{Pop}} \; -\ln P_\Phi(\tilde{z}_\Phi(y)) + \lambda \mathrm{Dist}(y, y_\Phi(\tilde{z}_\Phi(y)))$$

Alternatively we have

$$\begin{aligned}
\mathrm{Dist}(y,\hat{y}) &= ||y-\hat{y}||_1 \qquad\qquad (L_1) \\
&= -\frac{1}{\lambda}\ln p(y|\hat{y}) + C \;\; \text{for } p(y|\hat{y}) \propto \exp(-\lambda||y-\hat{y}||_1)
\end{aligned}$$

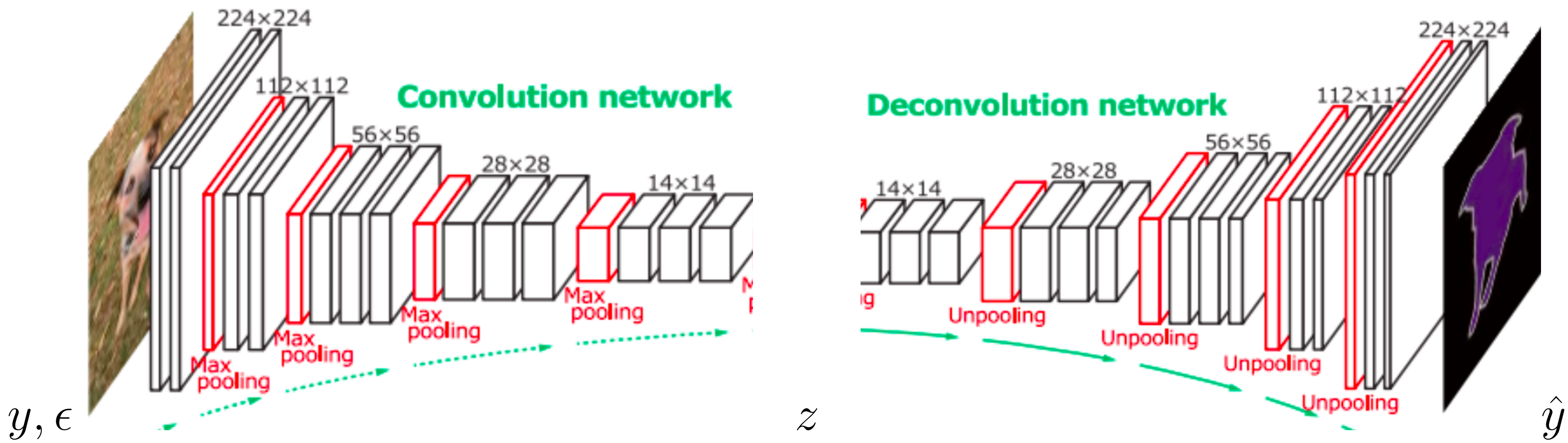
Again, we will ignore the log density interpretation and just call this L_1 distortion.

A Variational Bound on Mutual Information

$$
\begin{aligned}
I(y,z) &= E_{y,\epsilon}\ \ln \frac{p_\Phi(z|y)}{p_\Phi(z)} \\
&= E_{y,\epsilon}\ \ln \frac{p_\Phi(z|y)}{\hat{p}_\Phi(z)} + E_{y,\epsilon}\ \ln \frac{\hat{p}_\Phi(z)}{p_\Phi(z)} \\
&= E_{y,\epsilon}\ \ln \frac{p_\Phi(z|y)}{\hat{p}_\Phi(z)} - KL(p_\Phi(z), \hat{p}_\Phi(z)) \\
&\leq E_{y,\epsilon}\ \ln \frac{p_\Phi(z|y)}{\hat{p}_\Phi(z)}
\end{aligned}
$$

The Noisy Channel RDA

$$\Phi^* \;=\; \underset{\Phi}{\operatorname{argmin}}\, E_{y,\epsilon}\;\ln\frac{p_\Phi(z_\Phi(y,\epsilon)|y)}{\hat{p}_\Phi(z_\Phi(y,\epsilon))} + \lambda \mathrm{Dist}(y, y_\Phi(z_\Phi(y,\epsilon)))$$



VAE = RDA

$$\text{VAE:} \quad \Phi^* = \operatorname*{argmin}_{\Phi} E_{y \sim \mathrm{Pop},\, z \sim \hat{P}_\Phi(z|y)} \;\; \ln \frac{\hat{P}_\Phi(z|y)}{P_\Phi(z)} - \ln P_\Phi(y|z)$$

$P_\Phi(z)$, $P_\Phi(y|z)$ and $\hat{P}_\Phi(z|y)$ are model components and we can switch the notation to $\hat{P}_\Phi(z)$ $\hat{P}_\Phi(y|z)$ and $P_\Phi(z|y)$ with no change in the model.

$$\text{RDA:} \quad \Phi^* = \operatorname*{argmin}_{\Phi} \; E_{y \sim \mathrm{Pop},\, z \sim P_\Phi(z|y)} \;\; \ln \frac{P_\Phi(z|y)}{\hat{P}_\Phi(z)} - \ln \hat{P}_\Phi(y|z)$$

In an RDA we take $P_\Phi(y, z)$ to be $\mathrm{Pop}(y) P_\Phi(z|y)$ so that the rate term is an upper bound on $I_\Phi(y, z)$.

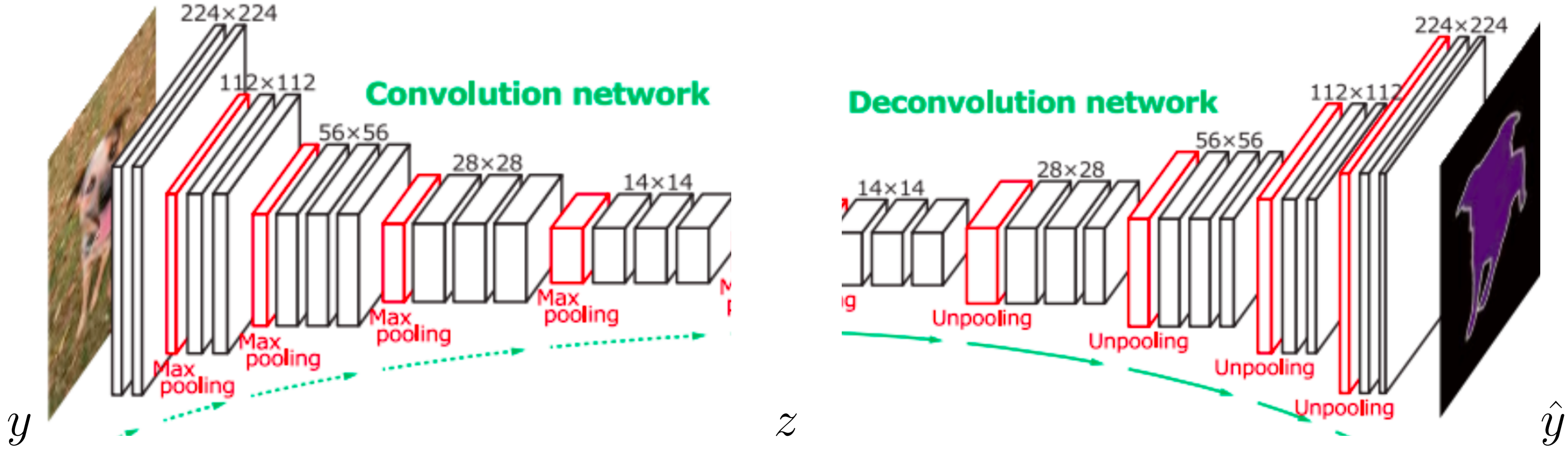
Sampling

We can require $\hat{p}_\Phi(z)$ be Gaussian. In that case we can sample z from $\hat{p}_\Phi(z)$ and generate images (as in a GAN).

[Alec Radford]

This is **sampling** — not compression. We are decompressing noise.

A General Autoencoder



We show below that for $p_\Phi(z|y)$ and $\hat{p}_\Phi(z)$ both required to be Gaussian we can assume without loss of generality that

$$\hat{p}_\Phi(z) = \mathcal{N}(0, I)$$

Gaussian Noisy-Channel RDA

We now show that a reparameterization can always convert $\hat{p}_\Phi(z)$ to a zero-mean identity-covariance Gaussian.

$$\Phi^* = \underset{\Phi}{\mathrm{argmin}}\, E_{y,\epsilon}\ \ln \frac{p_\Phi(z_\Phi(y,\epsilon)|y)}{\hat{p}_\Phi(z_\Phi(y,\epsilon))} + \lambda \mathrm{Dist}(y, y_\Phi(z_\Phi(y,\epsilon)))$$

$$\begin{aligned}
z_\Phi(y,\epsilon) &= \mu_\Phi(y) + \sigma_\Phi(y) \odot \epsilon \quad \epsilon \sim \mathcal{N}(0, I) \\
p_\Phi(z[i]|y) &= \mathcal{N}(\mu_\Phi(y)[i], \sigma_\Phi(y)[i])) \\
\hat{p}_\Phi(z[i]) &= \mathcal{N}(\hat{\mu}_z[i], \hat{\sigma}_z[i]) \\
\mathrm{Dist}(y, \hat{y}) &= ||y - \hat{y}||^2
\end{aligned}$$

Gaussian Noisy-Channel RDA

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}}\, E_{y,\epsilon}\ \ln \frac{p_\Phi(z_\Phi(y,\epsilon)|y)}{\hat{p}_\Phi(z_\Phi(y,\epsilon))} + \lambda \mathrm{Dist}(y, y_\Phi(z_\Phi(y,\epsilon)))$$

We will show that we can fix $\hat{p}_\Phi(z)$ to $\mathcal{N}(0, I)$.

$$\begin{aligned} p_\Phi(z[i]|y) &= \mathcal{N}(\mu_\Phi(y)[i], \sigma_\Phi(y)[i]) \\ \hat{p}_\Phi(z[i]) &= \mathcal{N}(0,1) \\ \mathrm{Dist}(y,\hat{y}) &= ||y-\hat{y}||^2 \end{aligned}$$

Gaussian Noisy-Channel RDA

$$\Phi^* \;=\; \underset{\Phi}{\operatorname{argmin}}\, E_{y,\epsilon}\;\ln \frac{p_\Phi(z_\Phi(y,\epsilon)|y)}{\hat{p}_\Phi(z_\Phi(y,\epsilon))} + \lambda \mathrm{Dist}(y, y_\Phi(z_\Phi(y,\epsilon)))$$

$$= \underset{\Phi}{\operatorname{argmin}}\; E_{y\sim\mathrm{Pop}} \left(\begin{array}{c} KL(p_\Phi(z|y), \hat{p}_\Phi(z)) \\ \\ +\lambda\; E_\epsilon\; \mathrm{Dist}(y,\; y_\Phi(z_\Phi(y,\epsilon))) \end{array} \right)$$

Closed Form KL-Divergence

$$KL(p_\Phi(z|y), \hat{p}_\Phi(z))$$

$$= \sum_i \frac{\sigma_\Phi(y)[i]^2 + (\mu_\Phi(y)[i] - \mu_z[i])^2}{2\sigma_z[i]^2} + \ln\frac{\sigma_z[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2}$$

Standardizing $\hat{p}_\Phi(z)$

$$KL(p_\Phi(z|y), p_\Phi(z))$$

$$= \sum_i \frac{\sigma_\Phi(y)[i]^2 + (\mu_\Phi(y)[i] - \mu_z[i])^2}{2\sigma_z[i]^2} + \ln \frac{\sigma_z[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2}$$

$$KL(p_{\Phi'}(z|y), \mathcal{N}(0, I))$$

$$= \sum_i \frac{\sigma_{\Phi'}(y)[i]^2 + \mu_{\Phi'}(y)[i]^2}{2} + \ln \frac{1}{\sigma_{\Phi'}(y)[i]} - \frac{1}{2}$$

Standardizing $\hat{p}_\Phi(z)$

$$
\begin{aligned}
KL_\Phi &= \sum_i \frac{\sigma_\Phi(y)[i]^2 + (\mu_\Phi(y)[i] - \mu_z[i])^2}{2\sigma_z[i]^2} + \ln\frac{\sigma_z[i]}{\sigma_\Phi(y)[i]} - \frac{1}{2} \\
KL_{\Phi'} &= \sum_i \frac{\sigma_{\Phi'}(y)[i]^2 + \mu_{\Phi'}(y)[i]^2}{2} + \ln\frac{1}{\sigma_{\Phi'}(y)[i]} - \frac{1}{2}
\end{aligned}
$$

Setting Φ' so that

$$
\begin{aligned}
\mu_{\Phi'}(y)[i] &= (\mu_\Phi(y)[i] - \mu_z[i])/\sigma_z[i] \\
\sigma_{\Phi'}(y)[i] &= \sigma_\Phi(y)[i]/\sigma_z[i]
\end{aligned}
$$

gives $KL(p_\Phi(z|y), \hat{p}_\Phi(z)) = KL(p_{\Phi'}(z|y), \mathcal{N}(0, I))$.

END