## TTIC 31230 Fundamentals of Deep Learning Quiz 1

**Problem 1: Optimizing Cross Entropy.** For this problem we consider a population distribution Pop on the non-negative natural numbers  $k \geq 0$ . We will work with the population mean  $\mu = E_{k \sim \text{Pop}} k$ . We consider model distributions defined by the single parameter  $\lambda$  with  $0 \leq \lambda < 1$  and defined by the distribution

$$Q_{\lambda}(k) = (1 - \lambda)\lambda^k$$

- (a) Given an expression for the cross-entropy  $H(\text{Pop}, Q_{\lambda})$  in terms of  $\mu$  and  $\lambda$ .
- (b) Solve for the optimal value  $\lambda^*$  minimizing  $H(\text{Pop}, Q_{\lambda})$  as a function of  $\mu$ .
- (c) Solve for mean value of the distribution  $Q_{\lambda^*}$  in terms of  $\mu$ .

**Problem 2. Maximum Mutual Information Training.** Consider a population distribution Pop on pairs  $\langle x, y \rangle$  and a model distribution  $Q_{\Phi}(\hat{y}|x)$ . Consider a distribution  $P_{\Phi}$  on triples  $x, y, \hat{y}$  where  $\langle x, y \rangle$  is drawn from Pop and  $\hat{y}$  is drawn from  $Q_{\Phi}(\hat{y}|x)$ . Under the distribution  $P_{\Phi}$  the mutual information between y and  $\hat{y}$  is defined by

$$I_{\Phi}(y, \hat{y}) = KL(P_{\Phi}(y, \hat{y}), \operatorname{Pop}(y)P_{\Phi}(\hat{y}))$$

$$P_{\Phi}(y, \hat{y}) = \sum_{x} \operatorname{Pop}(x) \operatorname{Pop}(y|x) Q_{\Phi}(\hat{y}|x)$$

$$= E_{x \sim \operatorname{Pop}} \operatorname{Pop}(y|x) Q_{\Phi}(\hat{y}|x)$$

$$\operatorname{Pop}(y) = E_{x \sim \operatorname{Pop}} \operatorname{Pop}(y|x)$$

$$P_{\Phi}(\hat{y}) = E_{x \sim \operatorname{Pop}} Q_{\Phi}(y|x)$$

Here we are interested in comparing the fundamental cross entropy objective to the objective of maximizing the mutual information  $P_{\Phi}(y, \hat{y})$ .

$$\begin{split} &\Phi_1^* &= & \underset{\Phi}{\operatorname{argmin}} E_{\langle x, \, y \rangle \sim \operatorname{Pop}} \, - \ln Q_{\Phi}(y|x) \\ &\Phi_2^* &= & \underset{\Phi}{\operatorname{argmax}} I_{\Phi}(y, \hat{y}) \end{split}$$

- (a) Suppose that there exists a perfect predictor a parameter setting  $\Phi^*$  such that  $P_{\Phi}(\hat{y}|x) = 1$  for  $\hat{y} = y$  and zero otherwise. Show using an explicit calculation and standard information theoretic inequalities that a perfect predictor is an optimum of both the cross-entropy objective and the maximum mutual information objective.
- (b) Consider binary classification where we have  $y, \hat{y} \in \{-1, 1\}$ . Show using an explicit calculation and standard information-theoretic inequalities that a perfect anti-predictor with  $P_{\Phi}(hayy|x) = 1$  for  $\hat{y} = -y$  is also optimal for the maximum mutual information objective.

**Problem 3. Backpropagation for Layer Normalization.** Layer normalization is an alternative to batch normalization and is used in the transformer to handle "covariate shift". In the transformer each a layer has positions in the text that I will index by t and neurons at each position that I will index by t. We can think of this as a sequence of vectors L[t, I]. Layer normalization is defined by the following equations where the vectors  $A_{\ell+1}[I]$  and  $B_{\ell+1}[I]$  are trained parameters and  $\sigma$  is an arbitrary activation function, typically ReLU.

$$\mu_{\ell} = \frac{1}{TI} \sum_{t,i} L_{\ell}[t,i]$$

$$\sigma_{\ell} = \sqrt{\frac{1}{TI} \sum_{t,i} (L_{\ell}[t,i] - \mu_{\ell})^2}$$

$$\tilde{L}_{\ell+1}[t,i] = \sigma \left(\frac{A_{\ell+1}[i]}{\sigma_{\ell}} (L_{\ell}[t,i] - \mu_{\ell}) + B_{\ell+1}[i]\right)$$

Write backpropagation equations for these three assignments.